Generalizing the 15-Puzzle Using Group Theory: Exploring D-Dimensional Puzzles with Unequal Sides and Multiple Empty Blocks

Qihe Yang

Zhengzhou Foreign Language School, Zhengzhou, China

yqhkevin2008@icloud.com

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Abstract: This paper conducts an in - depth study and generalization of the 15 - puzzle using group - theoretic methods. Starting with two - dimensional puzzles, relevant concepts are defined in detail, and the solvability of Lloyd's Puzzle, n×n, and n×m puzzles is studied by constructing permutations. The research is extended to three - dimensions, where multiple puzzle types are defined, and the solvability conditions of n^3, n^2m, and nmk puzzles are obtained. Further generalization to d - dimensions is carried out, with the definitions and solvability conclusions of n^d and s -sliced n^d puzzles given. The characteristics of puzzles with multiple empty blocks and generalized initial configurations are also explored. The research results deepen the understanding of the 15 - puzzle and its extended forms, providing a theoretical basis for subsequent related research.

1. Introduction

The "15 - puzzle" is a captivating combinatorial puzzle with a rich historical backdrop. Dating back to 1874, it was ingeniously invented by Noyes Chapman and swiftly gained remarkable popularity during the 1880s. This simple yet challenging puzzle consists of 15 numbered square blocks placed in a 4x4 grid, with one empty space, inviting players to arrange the blocks in ascending order through a series of sliding moves. Subsequently, Sam Lloyd added an exciting twist to the puzzle - solving landscape by offering a substantial $1000 prize to anyone who could crack his unique version, known as Lloyd's Puzzle. However, in 1879, Johnson and Story [1] demonstrated the futility of this pursuit, proving that Lloyd's Puzzle was, in fact, unsolvable. This revelation did not dampen the enthusiasm of the academic community; instead, it sparked a flurry of research endeavors aimed at further exploring the 15 - puzzle and its generalizations. Over the years, scholars from diverse fields have employed an array of mathematical tools to delve into the properties of these puzzles. [2] Archer, in 1999 [3], utilized graph theory to analyze the puzzle's state - space structure, representing each configuration as a vertex and each legal move as an edge in a graph. This approach provided valuable insights into the connectivity and reachability of different puzzle states. [4] In 2019, Chu and Hough brought a probabilistic perspective to the table, exploring the likelihood of reaching a solved state from a given initial configuration. Their work shed light on the statistical behavior of the puzzle - solving process. More recently, in 2023, Beyer, Mereta, Roldan, and Voran presented a tangential solution using topology. They exploited topological concepts to classify and analyze puzzle configurations, uncovering deep - seated relationships between different states. The first two studies were limited to generalizing the puzzle to n × n grids, while the latter focused on 2n - puzzles. In this paper, we embark on a novel exploration by adopting a group - theoretic approach. Our aim is to push the boundaries of existing research by generalizing the puzzle to d dimensions. This generalization not only encompasses puzzles with equal - side lengths but also extends to those with unequal side lengths. Additionally, we delve into puzzles containing multiple empty blocks, a hitherto less - explored aspect of the 15 - puzzle family. By doing so, we seek to uncover new theoretical insights and establish a more comprehensive understanding of the fundamental properties of these puzzles.

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2. Proofs for 2 - Dimensional Puzzles

2.1 2-dimensional Puzzles

Figure 1: A Sample Puzzle

In this paper, we denote a block by simply refering to its value; we denote a position by saying ”position”, followed by the value of the block that occupies that place in the solved state. For example, in Fig 1, we say that 6 is position 11. We will refer to puzzles by the lengths of the edges of the puzzle; we will also refer to two dimensional puzzles with explicit ”×” for both numerical and algebraic edge lengths, but omit ”×” when referring to higher dimensional puzzles with algebraic edge length. For example, a 2-dimensional puzzle with edge lengths 10 and 33 will be referred to as a 10×33 puzzle, and a 4-dimensional puzzle with edge lengths 4, 7, 9 and 11 will be referred to as a 4 × 7 × 9 × 11 puzzle; a 2-dimensional puzzle with edge lengths a and b will be referred to as a a × b puzzle ,and a 4-dimensional puzzle with edge lengths a, b, c and d will be referred to as a abcd puzzle.

2.2 Lloyd’s Puzzle and n × n Puzzles

To solve the puzzle is to make the blocks be in the correct position. We can take the solved configuration of the puzzle, and the initial configuration of the puzzle, and construct a permutation. For example with the puzzle in Figure 2, we obtain relationship 1

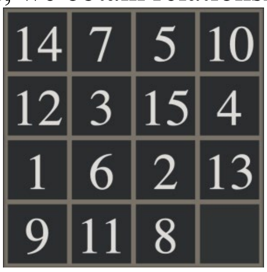

Figure 2: An random configuration

$$A = \begin{pmatrix} 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16 \\ 14\ 7\ 5\ 10\ 12\ 3\ 15\ 4\ 1\ 6\ 2\ 13\ 9\ 11\ 8\ 16 \end{pmatrix}$$

$$A = (1\ 14\ 11\ 2\ 7\ 15\ 8\ 4\ 10\ 6\ 3\ 5\ 12\ 13\ 9)\ (16) \tag{1}$$

And with the puzzle in the Figure 3 below, we obtain relationship 2

Figure 3: Another random configuration

$$A = \begin{pmatrix} 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16 \\ 15\ 1\ 12\ 5\ 7\ 11\ 4\ 8\ 10\ 6\ 3\ 14\ 9\ 2\ 13\ 16 \end{pmatrix}$$

$$A = (1\ 15\ 13\ 9\ 10\ 6\ 11\ 3\ 12\ 14\ 2)\ (4\ 5\ 7)\ (8)\ (16) \tag{2}$$

The benefit of this process is that it shows exactly what transpositions must take place for the initial configuration to be turned into the solved configuration, or what transpositions are required to solve the puzzle. However, observing the mechanics of the puzzle, we find that generally, two blocks cannot simply transpose. How a block actually moves is by sliding the block into the empty tile, which has the value 16. That is to say that an operation on the board to turn one configuration to another is a transposition with 16, or a composition of transpositions with 16.

Theorem 1. A configuration is a group under a move.

Proof. Associativity: (a b) (b c) (c d) = (a b c) (c d) = (a b) (b c d) = (a b c d) ; Identity: a move with itself; Inverse: the inverse of $(a\ b)$ is $(b\ a)$.

Every transposition in the permutation is, in this way, mediated by 16. So to see if these transpositions are possible, we can focus on 16. to see if the transpositions are possible, is the same as checking if the puzzle is solvable, since if the transpositions are not possible, the blocks cannot be moved into the correct places.

Lloyd's rendition of the puzzle has the 14 and 15 blocks transposed, while keeping all the other blocks in their solved state position. For Lloyd's question, where we get the permutation below.

$$A = \begin{pmatrix} 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16 \\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16 \end{pmatrix}$$

$$A = (1)\,(2)\,(3)\,(4)\,(5)\,(6)\,(7)\,(8)\,(9)\,(10)\,(11)\,(12)\,(13)\,(14\ 15)\,(16) \qquad (3)$$

We get a single transposition of (14 15) (4)

We now examine 16. 16 stays in the same place in the initial configuration (which is the permutation above in 3), and in the solved configuration. Hence, 16 must have been moved up and down, righ and left the same amount of times, which is an even number of transpositions. However, with the configuration above, an odd number of transpositions must occur in order for it to be solved. This means that it is impossible for this configuration to be changed into the solved configuration, or, this configuration is not solvable. Similarly, all puzzles that have this contradiction are not solvable. Thus we claim

Lemma 1. If 16 is in the same position at the start and in the end, there must be an even number of transpositions for the puzzle to be solvable.

However, counting transpositions from permutations is generally very cumbersome and inefficient, so we shall examine the number of permutations.

For a 4 × 4 puzzle, in order to get an even number of transpositions, one would need an even number of permutations. In the solved state, there are 16 permutations, which is an even number. For each 2 transpositions added, one could either combine 3 permutations to one, for example

$$(6)\,(7)\,(8) \rightarrow (6\ 7\ 8) \qquad (5)$$

Or combine 4 permutations to 2,

$$(6)\,(7)\,(8)\,(9) \rightarrow (6\ 7)\,(8\ 9) \qquad (6)$$

Or more generally

$$(3\ 4\ 6)\,(2\ 8)\,(1) \rightarrow (1\ 2\ 3\ 4\ 6\ 8) \qquad (7)$$

$$(3\ 4\ 6)\,(2\ 8)\,(1)\,(9\ 11\ 12) \rightarrow (1\ 9\ 11\ 12)\,(2\ 3\ 4\ 6\ 8) \qquad (8)$$

Both of which results in an even change in permutations.

Lemma 2. Combining any 3 permutations to 1, or combing any 4 permutations

To 2 will always increase the number of transposition by 2.

Proof. A permutation with x elements has $x - 1$ transpositions. Consider 3 permutations with a, b and c elements respectively. The total number of transpositions of the three permutations is thus $n_1 = a+b+c-3$. When we combine the 3 permutations, we obtain a single permuation that has $a + b + c$ elements. This new permutations thus has $n_2 = a+b+c-1$ transpositions. The change in the number of transpostions $n_2 - n_1 = 2$. Similarly, consider 4 permutations A, B, C and D with a, b, c and d elements respectively. The total number of the four transpositions of the permutations is $n_1 = a + b$

$+ c + d - 4$. Without the 4loss of generality, suppose we combine A and B together, and combine C and D together. When we combine the four permuations, we obtain 2 permuations: one with $a + b$ elements, and one with $c + d$ elements. They have $a + b - 1$ and $c + d - 1$ transpositions respectively, resulting in the total number of transpositions $n2 = a + b + c + d - 2$. The change in the number of transpositions $n_2 - n_1 = 2$. And thus completing the proof.

Theorem 2. An even change in the number of transpositions in a permutation group results in an even change in the number of permutations, and thus does not change the parity of the number of the permutations in the group.

This, combined with the even number of permutations initially, gives us an even number for the number of transpositions. From here, it is natural to attempt to extend this observation to $n \times n$ puzzles. All $n \times n$ puzzles are fundamentally the same

Lemma 3. If n 2 is in the same position at the start and in the end, there must be an even number of transpositions for the puzzle to be solvable.

According to previous logic. Here, we can categorise $n \times n$ puzzles into 2 subcategories, namely, where n is even, and where n is odd. Previously, we have discovered that for each 2 transpositions added, the parity of the number of permutations does not change; thus, the parity of the required number of permutations is entirely dependent on the parity of the number of permutations in the solved state. For n is even, it is even; and for n is odd, it is odd. Therefore, for n is even, the required parity of the number of permutations for the puzzle to be solvable is even, and for n is odd, it is then conversely odd. The examination of regular puzzles is now complete. But we can further generalise by noticing that $n \times n$ is in fact, a special case of $n \times m$, where $n = m$.

2.3 n × m Puzzles

For the sake of simplicity, we shall assume that $n \geq m$. A $n \times m$ puzzle has n rows and m columns, where we only have to arrange $n \times m - 1$ blocks in a similar fashion as the $n \times n$ puzzle. In fact, we can view the $n \times m$ puzzle as a section of $n \times n$ puzzle, or a sliced version along one direction. We can complement the $n \times m$, to a $n \times n$; not only physically as an aid in conception, but also numerically. When we attempt to solve a, for example, 4×4 puzzle, we first clear the first two rows, and then try to rearrange the lower 2 rows. This ”rearranging the lower 2 rows” is effectively solving a 4×2 puzzle. This way, we can consider the $n \times m$ puzzle as having $n - m$ rows above it that are already solved. This means that we can complement the puzzle numerically, by considering the above $n - m$ rows as having a $n(n - m)$ blocks that already in 5the correct position, thus having $n(n - m)$ permutations already. The benefit of complementing the puzzle like this is that we can use our conclusions of the $n \times n$ puzzles. Thus we can define a slicing, a slice size, a sliced configuration, a complementary configuration and a complemented configuration.

Lemma 4. A sliced configuration is a partially solved complemented configuration. It has additional permutations without transpositions of the size of the complementary configuration.

The total number of permutations of a complemented puzzle is simply the number of existing permutations (permutations of the actual puzzle), together with the number of permutations in the complementary part.

$$t = c + p \tag{9}$$

Where t is the number of permutations of the total, complemented puzzle, c is the number of permutations of the complementary part, p is the number of permutations of the actual puzzle. We are interested in the parity if all of these, since the parity determines if the puzzle is solvable or not. We can determine the required parity of t by n. It follows that

$$t \equiv c + p \pmod 2 \tag{10}$$

We can also determine the parity of c by n and m, here given by $c = n(n - m)$, so we can derive the equation that gives must be hold if the puzzle is solvable:

$$p \equiv t - c \pmod 2 \tag{11}$$

We shall omit (mod 2) in the future unless when it does not cause confusion. There are immediately four cases to this problem, we show these in Table 1 Now the examination of all 2-dimensional puzzles has been completed.

Table 1: All the Cases

Case	n	m	t	c	p
1	0	0	0	0	0
2	0	1	0	0	0
3	1	0	1	1	0
4	1	1	1	0	1

3. Proofs for 3 - Dimensional Puzzles

3.1 3 - Dimensional Puzzles

3-dimensional puzzles can be conceived of as a generalisation of the 2-dimensional puzzle. However, there are no existing formal construction or definition of such a 3-dimensional puzzle.

Analogous to the treatment of the 2-dimensional puzzles, we can construct a permutation group under a move, and examine the parity of transpositions and permutations.

Similarly, if we keep n^3 in the same position at the start and in the end, n^3 must have been transposed up and down the same number of times, and similarly left and right, to and fro. This means that it has been transposed an even number of times. Thus, an even number of transpositions is required for the puzzle to be solvable, similar to the result we obtained in lemma 1.

Lemma 5. If n 3 is in the same position at the start and in the end, there must be an even number of transpositions for the puzzle to be solvable.

The parity of the number of solved-state permutations is given by $x = n^3 (\text{mod } 2)$. According to Theorem 2, for every two transpositions added, the parity of the number of permutatations does not change. Thus parity of the number of permutations needed for the puzzle to be solvable is entirely dependent on *n*. Namely, if *n* is even, then the number must be even; if *n* is odd, then the number must be odd.

3.2 n^2m and nmk Puzzles

We can make a generalisation from n^3 puzzles to n^2m puzzles. Suppose $n > m$. The n^2m configuration is a sliced configuration of the n^3 puzzle, with slice size $n - m$. This means that the complementary configuration is a $n(n - m)$ configuration. According to lemma 4, we know that it has an additional n^2 $(n - m)$ permutations. There are of course 4 cases again, as shown in table 2.

Table 2: All the Cases

Case	n	m	t	c	p
1	0	0	0	0	0
2	0	1	0	0	0
3	1	0	1	1	0
4	1	1	1	0	1

Where $t = n^3$, $c = n^2$ $(n - m)$. The requirements of solvability are given in the *p* column in table 2.

For a 2-dimension puzzle, there is only one possible directions of slicing. However, for a 3-dimensional puzzle, there are two possible direction of slicing. The $8nmk$ puzzle is a n^3 puzzle sliced in two directions, or a n^2m puzzle sliced in the other direction. Thus we can treat the *nmk* puzzle as the sliced configuration, and the n^2m as the complemented configuration, with slice size $n-k$. The benefit of this conception is that we can use our conclusions about the n^2m puzzle as aid; however, we can also directly use our conclusions from the n^3 puzzle. We can obtain the following tables.

We can immediately see that although *t* and *c* in tables 3 and 4 are different, the results they gave for *p* are identical. Now we have completed the examination of all 3-dimensional puzzles.

Table 3: All the Case

Case	n	m	k	t	c	p
1	0	0	0	0	0	0
2	0	1	0	0	0	0
3	1	0	0	0	0	0
4	1	1	0	1	1	0
1	0	0	1	0	0	0
2	0	1	1	0	0	0
3	1	0	1	0	0	0
4	1	1	1	1	0	1

Where $t = n^2m$, $c = nm(n - k)$.

Table 4: All the Case

Case	n	m	k	t	c	p
1	0	0	0	0	0	0
2	0	1	0	0	0	0
3	1	0	0	1	1	0
4	1	1	0	1	1	0
1	0	0	1	0	0	0
2	0	1	1	0	0	0
3	1	0	1	1	1	0
4	1	1	1	1	0	1

Where $t = n^3$, $c = n\ 2\ (n - m) + nm(n - k)$.

4. Proofs for n - Dimensional Puzzles

4.1 n^d Puzzles

The d-dimensional puzzles are harder to conceive, but we can define a $n$dimensional puzzle in a the same fashion as the 3-dimensional puzzle.

Previously, we have made the generalisation from n^2 to $n \times m$, and from n^3 to n^2m and further to nmk; we treated the n^2 puzzle as a special case of $n \times m$ when $n = m$, and the n^3 puzzle as a special case or nmk when $n = m = k$, as they are the more general case. Thus, we will directly examine the generalised version of n-dimensional puzzles.

4.2 s-sliced n^d Puzzles

We have established lemmas 3 and 5. In the more general case, where we have

One empty block and a d dimensional puzzle, we have the following

Theorem 3. If the configuration only has one empty block, and if the block should be in the same position in the initial configuration and in the solved configuration, an even number of transpositions is required for the puzzle to be solvable.

We now investigate the parity of the number of permutations. However, with a s-sliced n^d puzzle, there are a total of 2^{s+1} cases, it is very inefficient to exhaust all the cases in a table. Notice, for the $n \times m$, the n^2m and the nmk puzzles, all the combinations of values of n, m, and k lead to $p = 0$, except for one case; specifically, when $n = m = 1$, or $n = m = k = 1$. In that case, $p = 1$. In other words, only when all the edge lengths are odd, p is required to be odd. If this were to be true for d-dimensions, it would tremendously simplify our process of determining the solvablity of a puzzle.

Theorem 4. For any s-sliced puzzle with slice sn having size mn, if and only if n = m1 = m2 = · · · = ms = 1, p = 1 is required for the puzzle to be solvable; otherwise, p = 0 is required.

5. Conclusion

This research uses group theory to deeply analyze the 15-puzzle and extends it to the d - dimensional space. It comprehensively explores the characteristics and solvability conditions of puzzles with different dimensions, side lengths, and numbers of empty blocks, achieving a series of key results. In the study of two - dimensional puzzles, a permutation relationship from the initial state to the target state is constructed, clarifying the crucial role of the empty block in the movement.Through the analysis of Lloyd's Puzzle, it is found that if the initial and end positions of the empty block are the same, a necessary condition for the puzzle to be solvable is that the number of transpositions is even. This conclusion is further extended to $n \times n$ puzzles, and the parity of the number of permutations required for the puzzle to be solvable is determined according to the parity of n. For $n \times m$ puzzles, they are regarded as a part of $n \times n$ puzzles. By means of complementation, a formula for the total number of permutations is obtained, and the conditions for the puzzle to be solvable are determined by dividing into different cases. In the research of three - dimensional puzzles, multiple puzzle types are defined, and the n^3 puzzle is defined as a specific stacking form. Similar to the analysis of two - dimensional puzzles, if the position of n^3 is the same at the start and the end, an even number of transpositions is required for the puzzle to be solvable, and the parity of the number of permutations when the puzzle is solvable depends on the parity of n. For n^2m and nmk puzzles, by using the concepts of slicing and complementary configuration, corresponding solvability conclusions are obtained, and the results of p in the solvability conditions are consistent under different representation methods. In the research of d - dimensional puzzles, the n^d puzzle and the s - sliced n^d puzzle are defined. It is proven that when there is a single empty block and its position remains unchanged, an even number of transpositions is required for the puzzle to be solvable, and a concise judgment condition for the solvability of the s - sliced puzzle is given: if and only if $n = m_1 = m_2 = \dots = m_s = 1$, $p = 1$ is required for the puzzle to be solvable; otherwise, $p = 0$. For puzzles with multiple empty blocks, the conjecture that "puzzles with more than one empty block are always solvable" is proposed and proven. In the research of puzzles with generalized initial configurations, the equivalence relationships of regular puzzles and proper incomplete puzzles are clarified, and the conclusion that the parity of the number of transpositions required must be the same as the parity of the number of transpositions of the empty block is obtained. Based on this, the required values of p for the solvability of s - sliced puzzles under different conditions are given. The results of this research not only deepen the theoretical understanding of the 15 - puzzle and its extended forms but also lay a solid foundation for subsequent research. In the future, puzzles with more constraints or rules can be explored to study their solvability and solution algorithms; parallel solution algorithms can be designed to improve the solution efficiency; and machine learning methods can be combined to predict the solvability of puzzles or optimize the solution strategy, further expanding the research boundaries of this field.

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